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## On a class of block operator matrices in system theory.

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# On a class of block operator matrices in system theory.

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**Abstract.** We consider a class of block operator matrices arising in the study of scattering passive systems, especially in the context of boundary control problems. We prove that these block operator matrices are indeed a subclass of block operator matrices considered in [Trostorff: A characterization of boundary conditions yielding maximal monotone operators. *J. Funct. Anal.*, 267(8): 2787–2822, 2014], which can be characterized in terms of an associated boundary relation.

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# 1 Introduction

Following [5], a natural class of  $C_0$ -semigroup generators  $-A$  arising in the context of scattering passive systems in system theory, can be described as a block operator matrix of the following form: Let  $E_0, E, H, U$  be Hilbert spaces with  $E_0 \subseteq E$  dense and continuous and let  $L \in L(E_0, H)$ ,  $K \in L(E_0, U)$ . Moreover, denote by  $L^\diamond \in L(H, E'_0)$  and  $K^\diamond \in L(U, E'_0)$  the dual operators of  $L$  and  $K$ , respectively, where we identify  $H$  and  $U$  with their dual spaces. Then  $A$  is a restriction of  $\begin{pmatrix} K^\diamond K & -L^\diamond \\ L & 0 \end{pmatrix}$  with domain

$$\mathcal{D}(A) := \{(u, w) \in E_0 \times H \mid K^\diamond K u - L^\diamond w \in E\}, \quad (1)$$

where we consider  $E \cong E'$  as a subspace of  $E'_0$ . It is proved in [5, Theorem 1.4] that for such operator matrices  $A$ , the operator  $-A$  generates a contractive  $C_0$ -semigroup on  $E \oplus H$  and a so-called scattering passive system, containing  $-A$  as the generator of the corresponding system node, is considered (see [4] for the notion of system nodes and scattering passive systems). This class of semigroup generators were particularly used to study boundary control systems, see e.g. [8, 9, 7]. In these cases,  $L$  is a suitable realization of a differential operator and  $K$  is a trace operator associated with  $L$ . More precisely,  $G_0 \subseteq L \subseteq G$ , where  $G_0$  and  $G$  are both densely defined closed linear operators, such that  $K|_{\mathcal{D}(G_0)} = 0$  (as a typical example take  $G_0$  and  $G$  as the realizations of the gradient on  $L_2(\Omega)$  for some open set  $\Omega \subseteq \mathbb{R}^n$  with  $\mathcal{D}(G_0) = H_0^1(\Omega)$  and  $\mathcal{D}(G) = H^1(\Omega)$ ). It turns out that in this situation, the operator  $A$  is a restriction of the operator matrix  $\begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix}$ , where  $D := -(G_0)^*$  (see Lemma 3.1 below). Such restrictions were considered by the author in [6], where it was shown that such (also nonlinear) restrictions are maximal monotone (a hence,  $-A$  generates a possibly nonlinear contraction semigroup), if and only if an associated boundary relation on the so-called boundary data space of  $G$  is maximal monotone.

In this note, we characterize the class of boundary relations, such that the corresponding operator  $A$  satisfies (1) for some Hilbert spaces  $E_0, U$  and operators  $L \in L(E_0, H)$ ,  $K \in L(E_0, U)$ . We hope that this result yields a better understanding of the semigroup generators used in boundary control systems and provides a possible way to generalize known system-theoretical results to a class of nonlinear problems.

The article is structured as follows. In Section 2 we recall the basic notion of maximal monotone relations, we state the characterization result of [6] and introduce the class of block operator matrices considered in [5]. Section 3 is devoted to the main result (Theorem 3.2) and its proof.

Throughout, every Hilbert space is assumed to be complex, its inner product  $\langle \cdot | \cdot \rangle$  is linear in the second and conjugate linear in the first argument and the induced norm is denoted by  $|\cdot|$ .

## 2 Preliminaries

### 2.1 Maximal monotone relations

In this section we introduce the basic notions for maximal monotone relations. Throughout let  $H$  be a Hilbert space.

**Definition.** Let  $C \subseteq H \oplus H$ . We call  $C$  *linear*, if  $C$  is a linear subspace of  $H \oplus H$ . Moreover, we define for  $M, N \subseteq H$  the *pre-set of  $M$  under  $C$*  by

$$[M]C := \{x \in H \mid \exists y \in M : (x, y) \in C\}$$

and the *post-set of  $N$  under  $C$*  by

$$C[N] := \{y \in H \mid \exists x \in N : (x, y) \in C\}.$$

The *inverse relation*  $C^{-1}$  of  $C$  is defined by

$$C^{-1} := \{(v, u) \in H \oplus H \mid (u, v) \in C\}.$$

A relation  $C$  is called *monotone*, if for each  $(x, y), (u, v) \in C$ :

$$\Re \langle x - u \mid y - v \rangle \geq 0.$$

A monotone relation  $C$  is called *maximal monotone*, if for each monotone relation  $B \subseteq H \oplus H$  with  $C \subseteq B$  we have  $C = B$ . Moreover, we define the *adjoint relation*  $C^* \subseteq H \oplus H$  of  $C$  by

$$C^* := \{(v, -u) \in H \oplus H \mid (u, v) \in C\}^\perp,$$

where the orthogonal complement is taken in  $H \oplus H$ . A relation  $C$  is called *selfadjoint*, if  $C = C^*$ .

*Remark 2.1.*

(a) A pair  $(x, y) \in H \oplus H$  belongs to  $C^*$  if and only if for each  $(u, v) \in C$  we have

$$\langle v \mid x \rangle_H = \langle u \mid y \rangle_H.$$

Thus, the definition of  $C^*$  coincides with the usual definition of the adjoint operator for a densely defined linear operator  $C : \mathcal{D}(C) \subseteq H \rightarrow H$ .

(b) Note that a selfadjoint relation is linear and closed, since it is an orthogonal complement.

We recall the famous characterization result for maximal monotone relations due to G. Minty.

**Theorem 2.2** ([2]). *Let  $C \subseteq H \oplus H$  be monotone. Then the following statements are equivalent:*

- (i)  $C$  is maximal monotone,
- (ii)  $\exists \lambda > 0 : (1 + \lambda C)[H] = H$ , where  $(1 + \lambda C) := \{(u, u + \lambda v) \mid (u, v) \in C\}$ ,
- (iii)  $\forall \lambda > 0 : (1 + \lambda C)[H] = H$ .

*Remark 2.3.*

(a) We note that for a monotone relation  $C \subseteq H \oplus H$  the relations  $(1 + \lambda C)^{-1}$  for  $\lambda > 0$  are Lipschitz-continuous mappings with best Lipschitz-constant less than or equal to 1. By the latter Theorem, maximal monotone relations are precisely those monotone relations, where  $(1 + \lambda C)^{-1}$  for  $\lambda > 0$  is defined on the whole Hilbert space  $H$ .

- (b) If  $C \subseteq H \oplus H$  is closed and linear, then  $C$  is maximal monotone if and only if  $C$  and  $C^*$  are monotone. Indeed, if  $C$  is maximal monotone, then  $(1 + \lambda C)^{-1} \in L(H)$  for each  $\lambda > 0$  with  $\sup_{\lambda > 0} \|(1 + \lambda C)^{-1}\| \leq 1$ . Hence,  $(1 + \lambda C^*)^{-1} = \left((1 + \lambda C)^{-1}\right)^* \in L(H)$  for each  $\lambda > 0$  with  $\sup_{\lambda > 0} \|(1 + \lambda C^*)^{-1}\| \leq 1$ . The latter gives for each  $(x, y) \in C^*$  and  $\lambda > 0$

$$\begin{aligned} |x + \lambda y|_H^2 &= |x|_H^2 + 2\Re \lambda \langle x | y \rangle_H + \lambda^2 |y|_H^2 \\ &= |(1 + \lambda C^*)^{-1}(x + \lambda y)|_H^2 + 2\Re \lambda \langle x | y \rangle_H + \lambda^2 |y|_H^2 \\ &\leq |x + \lambda y|_H^2 + 2\Re \lambda \langle x | y \rangle_H + \lambda^2 |y|_H^2 \end{aligned}$$

and hence,

$$-\frac{\lambda}{2}|y|^2 \leq \Re \langle x | y \rangle_H.$$

Letting  $\lambda$  tend to 0, we obtain the monotonicity of  $C^*$ . If on the other hand  $C$  and  $C^*$  are monotone, we have that  $[\{0\}](1 + \lambda C^*) = \{0\}$  for each  $\lambda > 0$  and thus,  $\overline{[H](1 + \lambda C)^{-1}} = \overline{(1 + \lambda C)[H]} = ([\{0\}](1 + \lambda C^*))^\perp = H$ . Since, moreover  $(1 + \lambda C)^{-1}$  is closed and Lipschitz-continuous due to the monotonicity of  $C$ , we obtain that  $[H](1 + \lambda C)^{-1}$  is closed, from which we derive the maximal monotonicity by Theorem 2.2.

## 2.2 Boundary data spaces and a class of maximal monotone block operator matrices

In this section we will recall the main result of [6]. For doing so, we need the following definitions. Throughout, let  $E, H$  be Hilbert spaces and  $G_0 : \mathcal{D}(G_0) \subseteq E \rightarrow H$  and  $D_0 : \mathcal{D}(D_0) \subseteq H \rightarrow E$  be two densely defined closed linear operators satisfying

$$G_0 \subseteq -(D_0)^*.$$

We set  $G := (-D_0)^* \supseteq G_0$  and  $D := -(G_0)^* \supseteq D_0$ , which are both densely defined closed linear operators.

**Example 2.4.** As a guiding example we consider the following operators. Let  $\Omega \subseteq \mathbb{R}^n$  open and define  $G_0$  as the closure of the operator

$$\begin{aligned} C_c^\infty(\Omega) &\subseteq L_2(\Omega) \rightarrow L_2(\Omega)^n \\ \phi &\mapsto (\partial_i \phi)_{i \in \{1, \dots, n\}}, \end{aligned}$$

where  $C_c^\infty(\Omega)$  denotes the set of infinitely differentiable functions compactly supported in  $\Omega$ . Moreover, let  $D_0$  be the closure of

$$\begin{aligned} C_c^\infty(\Omega) &\subseteq L_2(\Omega)^n \rightarrow L_2(\Omega) \\ (\phi_i)_{i \in \{1, \dots, n\}} &\mapsto \sum_{i=1}^n \partial_i \phi_i. \end{aligned}$$

Then, by integration by parts, we obtain  $G_0 \subseteq -(D_0)^*$ . Moreover, we have that  $G : \mathcal{D}(G) \subseteq L_2(\Omega) \rightarrow L_2(\Omega)^n$ ,  $u \mapsto \text{grad } u$  with  $\mathcal{D}(G) = H^1(\Omega)$  as well as  $D : \mathcal{D}(D) \subseteq L_2(\Omega)^n \rightarrow L_2(\Omega)$ ,  $v \mapsto \text{div } v$  with  $\mathcal{D}(D) = \{v \in L_2(\Omega)^n \mid \text{div } v \in L_2(\Omega)\}$ , where  $\text{grad } u$  and  $\text{div } v$  are meant in the sense of distributions. We remark that in case of a smooth boundary  $\partial\Omega$  of

$\Omega$ , elements  $u \in \mathcal{D}(G_0) = H_0^1(\Omega)$  are satisfying  $u = 0$  on  $\partial\Omega$  and elements  $v \in \mathcal{D}(D_0)$  are satisfying  $v \cdot n = 0$  on  $\partial\Omega$ , where  $n$  denotes the unit outward normal vector field. Thus,  $G_0$  and  $D_0$  are the gradient and divergence with vanishing boundary conditions, while  $G$  and  $D$  are the gradient and divergence without any boundary condition. In the same way one might treat the case of  $G_0 = \text{curl}_0$ , the rotation of vectorfields with vanishing tangential component and  $G = \text{curl}$ . Note that then  $D_0 = -\text{curl}_0$  and  $D = -\text{curl}$ .

As the previous example illustrates, we want to interpret  $G_0$  and  $D_0$  as abstract differential operators with vanishing boundary conditions, while  $G$  and  $D$  are the respective differential operators without any boundary condition. This motivates the following definition.

**Definition.** We define the spaces<sup>1</sup>

$$\mathcal{BD}(G) := (\mathcal{D}(G_0))^{\perp_{\mathcal{D}_G}}, \quad \mathcal{BD}(D) := (\mathcal{D}(D_0))^{\perp_{\mathcal{D}_D}},$$

where the orthogonal complements are taken in  $\mathcal{D}_G$  and  $\mathcal{D}_D$ , respectively. We call  $\mathcal{BD}(G)$  and  $\mathcal{BD}(D)$  *abstract boundary data spaces associated with  $G$  and  $D$* , respectively. Consequently, we can decompose  $\mathcal{D}_G = \mathcal{D}_{G_0} \oplus \mathcal{BD}(G)$  and  $\mathcal{D}_D = \mathcal{D}_{D_0} \oplus \mathcal{BD}(D)$ . We denote by  $\pi_{\mathcal{BD}(G)} : \mathcal{D}_G \rightarrow \mathcal{BD}(G)$  and by  $\pi_{\mathcal{BD}(D)} : \mathcal{D}_D \rightarrow \mathcal{BD}(D)$  the corresponding orthogonal projections. In consequence,  $\pi_{\mathcal{BD}(G)}^* : \mathcal{BD}(G) \rightarrow \mathcal{D}_G$  and  $\pi_{\mathcal{BD}(D)}^* : \mathcal{BD}(D) \rightarrow \mathcal{D}_D$  are the canonical embeddings and we set  $P_{\mathcal{BD}(G)} := \pi_{\mathcal{BD}(G)}^* \pi_{\mathcal{BD}(G)} : \mathcal{D}_G \rightarrow \mathcal{D}_G$  as well as  $P_{\mathcal{BD}(D)} := \pi_{\mathcal{BD}(D)}^* \pi_{\mathcal{BD}(D)} : \mathcal{D}_D \rightarrow \mathcal{D}_D$ . An easy computation gives

$$\mathcal{BD}(G) = [\{0\}](1 - DG), \quad \mathcal{BD}(D) = [\{0\}](1 - GD)$$

and thus,  $G[\mathcal{BD}(G)] \subseteq \mathcal{BD}(D)$  as well as  $D[\mathcal{BD}(D)] \subseteq \mathcal{BD}(G)$ . We set

$$\begin{aligned} \dot{G} : \mathcal{BD}(G) &\rightarrow \mathcal{BD}(D), x \mapsto Gx \\ \dot{D} : \mathcal{BD}(D) &\rightarrow \mathcal{BD}(G), x \mapsto Dx \end{aligned}$$

and observe that both are unitary operators satisfying  $(\dot{G})^* = \dot{D}$  (see [3, Section 5.2] for details).

Having these notions at hand, we are ready to state the main result of [6].

**Theorem 2.5** ([6, Theorem 3.1]). *Let  $G_0, D_0, G$  and  $D$  be as above and let*

$$A \subseteq \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} : \mathcal{D}(A) \subseteq H_0 \oplus H_1 \rightarrow H_0 \oplus H_1$$

*be a (possibly nonlinear) restriction of  $\begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} : \mathcal{D}(G) \times \mathcal{D}(D) \subseteq H_0 \oplus H_1 \rightarrow H_0 \oplus H_1, (u, w) \mapsto (Dw, Gu)$ . Then  $A$  is maximal monotone, if and only if there exists a maximal monotone relation  $h \subseteq \mathcal{BD}(G) \oplus \mathcal{BD}(D)$  such that*

$$\mathcal{D}(A) = \left\{ (u, w) \in \mathcal{D}(G) \times \mathcal{D}(D) \mid \left( \pi_{\mathcal{BD}(G)} u, \dot{D} \pi_{\mathcal{BD}(D)} w \right) \in h \right\}.$$

*We call  $h$  the boundary relation associated with  $A$ .*

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<sup>1</sup>For a closed linear operator  $C$  we denote by  $\mathcal{D}_C$  its domain, equipped with the graph-norm of  $C$ .



## 2.3 A class of block operator matrices in system theory

In [5] the following class of block operator matrices is considered: Let  $E, E_0, H, U$  be Hilbert spaces such that  $E_0 \subseteq E$  with dense and continuous embedding. Moreover, let  $L \in L(E_0, H)$  and  $K \in L(E_0, U)$  such that

$$\begin{pmatrix} L \\ K \end{pmatrix} : E_0 \subseteq E \rightarrow H \oplus U$$

is closed. This assumption particularly yields that the norm on  $E_0$  is equivalent to the graph norm of  $\begin{pmatrix} L \\ K \end{pmatrix}$ . We define  $L^\diamond \in L(H, E'_0)$  and  $K^\diamond \in L(U, E'_0)$  by  $(L^\diamond x)(w) := \langle x | Lw \rangle_H$  and  $(K^\diamond u)(w) := \langle u | Kw \rangle_U$  for  $x \in H, w \in E_0, u \in U$  and consider the following operator

$$A \subseteq \begin{pmatrix} K^\diamond K & -L^\diamond \\ L & 0 \end{pmatrix} : \mathcal{D}(A) \subseteq E \oplus H \rightarrow E \oplus H \quad (2)$$

with  $\mathcal{D}(A) := \{(u, w) \in E_0 \times H \mid K^\diamond K u - L^\diamond w \in E\}$ , where we consider  $E \cong E' \subseteq E'_0$  as a subspace of  $E'_0$ . We recall the following result from [5], which we present in a slight different formulation<sup>2</sup>.

**Theorem 2.6** ([5, Theorem 1.4]). *The operator  $A$  defined above is maximal monotone.*

*Remark 2.7.* We remark that in [5, Theorem 1.4] the operator  $-A$  is considered and it is proved that  $-A$  is the generator of a contraction semigroup.

We note that operators of the form (2) were applied to discuss boundary control problems. For instance in [9, 7] the setting was used to study the wave equation with boundary control on a smooth domain  $\Omega \subseteq \mathbb{R}^n$ . In this case the operator  $L$  was a suitable realization of the gradient on  $L_2(\Omega)$  and  $K$  was the Dirichlet trace operator. More recently, Maxwell's equations on a smooth domain  $\Omega \subseteq \mathbb{R}^3$  with boundary control were studied within this setting (see [8]). In this case  $L$  was a suitable realization of curl, while  $K$  was the trace operator mapping elements in  $\mathcal{D}(L)$  to their tangential component on the boundary.

In both cases, there exist two closed operators  $G_0 : \mathcal{D}(G_0) \subseteq E \rightarrow H$ ,  $D_0 : \mathcal{D}(D_0) \subseteq H \rightarrow E$  with  $G_0 \subseteq -(D_0)^* =: G$  such that  $G_0 \subseteq L \subseteq G$  and  $K|_{\mathcal{D}(G_0)} = 0$  (cp. Example 2.4). It is the purpose of this paper to show how the operators  $A$  in (2) and in Theorem 2.5 are related in this case.

## 3 Main result

Let  $E, H$  be Hilbert spaces and  $G_0 : \mathcal{D}(G_0) \subseteq E \rightarrow H$  and  $D_0 : \mathcal{D}(D_0) \subseteq H \rightarrow E$  be densely defined closed linear operators with  $G_0 \subseteq -(D_0)^* =: G$  and  $D_0 \subseteq -(G_0)^* =: D$ .

**Hypothesis.** We say that two Hilbert spaces  $E_0, U$  and two operators  $L \in L(E_0, H)$  and  $K \in L(E_0, U)$  satisfy the hypothesis, if

- (a)  $E_0 \subseteq E$  dense and continuous.

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<sup>2</sup>We note that in [5] an additional operator  $G \in L(E_0, E'_0)$  is incorporated in  $A$ , which we will omit for simplicity.

(b)  $\begin{pmatrix} L \\ K \end{pmatrix} : E_0 \subseteq E \rightarrow H \oplus U$  is closed.

(c)  $G_0 \subseteq L \subseteq G$  and  $K|_{\mathcal{D}(G_0)} = 0$ .

**Lemma 3.1.** *Assume that  $E_0, U$  and  $L, K$  satisfy the hypothesis. Let  $(u, w) \in E_0 \times H$  such that  $K^\diamond Ku - L^\diamond w \in E$ . Then  $w \in \mathcal{D}(D)$  and  $Dw = K^\diamond Ku - L^\diamond w$ .*

*Proof.* For  $v \in \mathcal{D}(G_0)$  we compute

$$\begin{aligned} \langle w | G_0 v \rangle_H &= \langle w | Lv \rangle_H \\ &= (L^\diamond w)(v) \\ &= (-K^\diamond Ku + L^\diamond w)(v) + (K^\diamond Ku)(v) \\ &= \langle -K^\diamond Ku + L^\diamond w | v \rangle_E + \langle Ku | Kv \rangle_U \\ &= \langle -K^\diamond Ku + L^\diamond w | v \rangle_E, \end{aligned}$$

where we have used  $G_0 \subseteq L$  and  $Kv = 0$ . The latter gives  $w \in \mathcal{D}(G_0^*) = \mathcal{D}(D)$  and  $Dw = -G_0^* w = K^\diamond Ku - L^\diamond w$ .  $\square$

The latter lemma shows that if the hypothesis holds and  $A$  is given as in (2), then  $A$  is a restriction of  $\begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix}$ , which, by Theorem 2.6, is maximal monotone. However, such restrictions are completely characterized by their associated boundary relation (see Theorem 2.5). The question, which now arises is: can we characterize those boundary relations, allowing to represent  $A$  as in (2)? The answer gives the following theorem.

**Theorem 3.2.** *Let  $A \subseteq \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix}$ . Then the following statements are equivalent.*

(i) *There exist Hilbert spaces  $E_0, U$  and operators  $L \in L(E_0, H)$ ,  $K \in L(E_0, U)$  satisfying the hypothesis, such that*

$$\mathcal{D}(A) = \{(u, w) \in E_0 \times H \mid K^\diamond Ku - L^\diamond w \in E\}.$$

(ii) *There exists  $h \subseteq \mathcal{BD}(G) \oplus \mathcal{BD}(G)$  maximal monotone and selfadjoint, such that*

$$\mathcal{D}(A) = \{(u, w) \in \mathcal{D}(G) \times \mathcal{D}(D) \mid (\pi_{\mathcal{BD}(G)} u, \overset{\bullet}{D} \pi_{\mathcal{BD}(D)} w) \in h\}.$$

We begin to prove the implication (i) $\Rightarrow$ (ii).

**Lemma 3.3.** *Assume (i) in Theorem 3.2 and set*

$$h := \left\{ (x, y) \in \mathcal{BD}(G) \oplus \mathcal{BD}(G) \mid \pi_{\mathcal{BD}(G)}^* x \in E_0, K^\diamond K \pi_{\mathcal{BD}(G)}^* x - L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y \in E \right\}.$$

*Then  $(u, w) \in \mathcal{D}(A)$  if and only if  $(u, w) \in \mathcal{D}(G) \times \mathcal{D}(D)$  with  $(\pi_{\mathcal{BD}(G)} u, \overset{\bullet}{D} \pi_{\mathcal{BD}(D)} w) \in h$ .*

*Proof.* Let  $(u, w) \in \mathcal{D}(A)$ . Then we know by Lemma 3.1, that  $(u, w) \in \mathcal{D}(G) \times \mathcal{D}(D)$ . We decompose  $u = u_0 + P_{\mathcal{BD}(G)}u$ , where  $u_0 \in \mathcal{D}(G_0) \subseteq E_0$ . Since  $u, u_0 \in E_0$  we get that  $\pi_{\mathcal{BD}(G)}^* \pi_{\mathcal{BD}(G)}u = P_{\mathcal{BD}(G)}u \in E_0$ . In the same way we decompose  $w = w_0 + P_{\mathcal{BD}(D)}w$ , where  $w_0 \in \mathcal{D}(D_0)$ . Since

$$(L^\diamond w_0)(z) = \langle w_0 | Lz \rangle_H = \langle w_0 | Gz \rangle_H = \langle -D_c w_0 | z \rangle_E$$

for each  $z \in E_0$ , we obtain  $L^\diamond w = -D_0 w_0 \in E$  and thus,

$$\begin{aligned} K^\diamond K \pi_{\mathcal{BD}(G)}^* \pi_{\mathcal{BD}(G)}u - L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} \overset{\bullet}{D} \pi_{\mathcal{BD}(D)}w &= K^\diamond K P_{\mathcal{BD}(G)}u - L^\diamond P_{\mathcal{BD}(D)}w \\ &= K^\diamond K(u - u_0) - L^\diamond(w - w_0) \\ &= K^\diamond Ku - L^\diamond w - D_c w_0 \in E, \end{aligned} \quad (3)$$

where we have used  $\overset{\bullet}{G} \overset{\bullet}{D} = 1$ ,  $Ku_0 = 0$  and  $(u, w) \in \mathcal{D}(A)$ . Thus, we have  $(\pi_{\mathcal{BD}(G)}u, \overset{\bullet}{D} \pi_{\mathcal{BD}(D)}w) \in h$ .

Assume now, that  $(u, w) \in \mathcal{D}(G) \times \mathcal{D}(D)$  with  $(\pi_{\mathcal{BD}(G)}u, \overset{\bullet}{D} \pi_{\mathcal{BD}(D)}w) \in h$ . Since  $u_0 := u - P_{\mathcal{BD}(G)}u \in \mathcal{D}(G_0) \subseteq E_0$  and by assumption  $P_{\mathcal{BD}(G)}u = \pi_{\mathcal{BD}(G)}^* \pi_{\mathcal{BD}(G)}u \in E_0$ , we infer that  $u \in E_0$ . Moreover, decomposing  $w = w_0 + P_{\mathcal{BD}(D)}w$  with  $w_0 \in \mathcal{D}(D_0)$  and using  $D_0 w_0 = -L^\diamond w_0$  we derive that

$$K^\diamond Ku - L^\diamond w = K^\diamond K \pi_{\mathcal{BD}(G)}^* \pi_{\mathcal{BD}(G)}u - L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} \overset{\bullet}{D} \pi_{\mathcal{BD}(D)}w + D_c w_0 \in E$$

by (3) and  $(\pi_{\mathcal{BD}(G)}u, \overset{\bullet}{D} \pi_{\mathcal{BD}(D)}w) \in h$ . Hence,  $(u, w) \in \mathcal{D}(A)$ .  $\square$

Although we already know that  $h$  in the previous Lemma is maximal monotone by Theorem 2.6 and Theorem 2.5, we will present a proof for this fact, which does not require these Theorems.

**Proposition 3.4.** *Assume (i) in Theorem 3.2 holds and let  $h \subseteq \mathcal{BD}(G) \oplus \mathcal{BD}(G)$  be as in Lemma 3.3. Then  $h$  is linear and maximal monotone.*

*Proof.* The linearity of  $h$  is clear due to the linearity of all operators involved. Let now  $(x, y) \in h$ . Then we compute

$$\begin{aligned} \Re \langle x | y \rangle_{\mathcal{BD}(G)} &= \Re \langle \pi_{\mathcal{BD}(G)}^* x | \pi_{\mathcal{BD}(G)}^* y \rangle_E + \Re \langle G \pi_{\mathcal{BD}(G)}^* x | G \pi_{\mathcal{BD}(G)}^* y \rangle_E \\ &= \Re \langle \pi_{\mathcal{BD}(G)}^* x | \pi_{\mathcal{BD}(G)}^* y \rangle_E + \Re \langle L \pi_{\mathcal{BD}(G)}^* x | G \pi_{\mathcal{BD}(G)}^* y \rangle_E \\ &= \Re \langle \pi_{\mathcal{BD}(G)}^* x | \pi_{\mathcal{BD}(G)}^* y \rangle_E + \Re \left( L^\diamond G \pi_{\mathcal{BD}(G)}^* y \right) (\pi_{\mathcal{BD}(G)}^* x) \\ &= \Re \left( L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y + \pi_{\mathcal{BD}(G)}^* y \right) (\pi_{\mathcal{BD}(G)}^* x) \\ &= \Re \left( L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y + \pi_{\mathcal{BD}(G)}^* y - K^\diamond K \pi_{\mathcal{BD}(G)}^* x \right) (\pi_{\mathcal{BD}(G)}^* x) + \\ &\quad + \langle K \pi_{\mathcal{BD}(G)}^* x | K \pi_{\mathcal{BD}(G)}^* x \rangle_U. \end{aligned}$$

Since  $\pi_{\mathcal{BD}(G)}^* x \in E_0$  and  $K^\diamond K \pi_{\mathcal{BD}(G)}^* x - L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y \in E$ , we get from Lemma 3.1  $K^\diamond K \pi_{\mathcal{BD}(G)}^* x - L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y = D \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y = \pi_{\mathcal{BD}(G)}^* y$ , since  $\overset{\bullet}{D} \overset{\bullet}{G} = 1$ . Thus, we obtain

$$\begin{aligned} \Re \langle x|y \rangle_{\mathcal{BD}(G)} &= \Re \left( L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y + \pi_{\mathcal{BD}(G)}^* y - K^\diamond K \pi_{\mathcal{BD}(G)}^* x \right) (\pi_{\mathcal{BD}(G)}^* x) + \\ &\quad + \langle K \pi_{\mathcal{BD}(G)}^* x | K \pi_{\mathcal{BD}(G)}^* x \rangle_U. \\ &= \langle K \pi_{\mathcal{BD}(G)}^* x | K \pi_{\mathcal{BD}(G)}^* x \rangle_U \geq 0. \end{aligned}$$

This proves the monotonicity of  $h$ . To show that  $h$  is maximal monotone, it suffices to prove  $(1+h)[\mathcal{BD}(G)] = \mathcal{BD}(G)$  by Theorem 2.2. Let  $f \in \mathcal{BD}(G)$  and consider the linear functional

$$E_0 \ni x \mapsto \langle \pi_{\mathcal{BD}(G)}^* f | x \rangle_E + \langle G \pi_{\mathcal{BD}(G)}^* f | Lx \rangle_H.$$

This functional is continuous and thus there is  $w \in E_0$  with<sup>3</sup>

$$\forall x \in E_0 : \langle w | x \rangle_E + \langle Lw | Lx \rangle_H + \langle Kw | Kx \rangle_U = \langle \pi_{\mathcal{BD}(G)}^* f | x \rangle_E + \langle G \pi_{\mathcal{BD}(G)}^* f | Lx \rangle_H. \quad (4)$$

In particular, for  $x \in \mathcal{D}(G_0) \subseteq E_0$  we obtain that

$$\begin{aligned} \langle Gw | G_c x \rangle_H &= \langle Lw | Lx \rangle_H \\ &= \langle \pi_{\mathcal{BD}(G)}^* f | x \rangle_E + \langle G \pi_{\mathcal{BD}(G)}^* f | Lx \rangle_H - \langle w | x \rangle_E \\ &= \langle \pi_{\mathcal{BD}(G)}^* f | x \rangle_E + \langle G \pi_{\mathcal{BD}(G)}^* f | G_c x \rangle_H - \langle w | x \rangle_E \\ &= \langle \pi_{\mathcal{BD}(G)}^* f | x \rangle_E - \langle DG \pi_{\mathcal{BD}(G)}^* f | x \rangle_E - \langle w | x \rangle_E \\ &= -\langle w | x \rangle_E, \end{aligned}$$

where we have used  $Kx = 0$  and  $DG \pi_{\mathcal{BD}(G)}^* f = \pi_{\mathcal{BD}(G)}^* f$ . The latter gives  $w \in \mathcal{D}(D)$  and  $DGw = w$  or, in other words,  $P_{\mathcal{BD}(G)} w = w$ . Set  $u := \pi_{\mathcal{BD}(G)}^* w$  and  $v = f - u$ . It is left to show that  $(u, v) \in h$ . First, note that  $\pi_{\mathcal{BD}(G)}^* u = P_{\mathcal{BD}(G)} w = w \in E_0$ . Moreover, we compute for  $x \in E_0$  using (4)

$$\begin{aligned} \left( K^\diamond K \pi_{\mathcal{BD}(G)}^* u - L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} v \right) (x) &= \langle K \pi_{\mathcal{BD}(G)}^* u | Kx \rangle_U - \langle \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} v | Lx \rangle_H \\ &= \langle Kw | Kx \rangle_U - \langle G \pi_{\mathcal{BD}(G)}^* f | Lx \rangle_H + \langle G \pi_{\mathcal{BD}(G)}^* u | Lx \rangle_H \\ &= \langle Kw | Kx \rangle_U - \langle G \pi_{\mathcal{BD}(G)}^* f | Lx \rangle_H + \langle Lw | Lx \rangle_H \\ &= \langle \pi_{\mathcal{BD}(G)}^* f | x \rangle_E - \langle w | x \rangle_E, \end{aligned}$$

which gives  $K^\diamond K \pi_{\mathcal{BD}(G)}^* u - L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} v = \pi_{\mathcal{BD}(G)}^* f - w \in E$ . This completes the proof.  $\square$

The only thing, which is left to show is that  $h$  is selfadjoint.

**Proposition 3.5.** *Assume (i) in Theorem 3.2 holds and let  $h \subseteq \mathcal{BD}(G) \oplus \mathcal{BD}(G)$  be as in Lemma 3.3. Then  $h$  is selfadjoint.*

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<sup>3</sup>Recall that the norm on  $E_0$  is equivalent to the graph norm of  $\begin{pmatrix} L \\ K \end{pmatrix}$ .

*Proof.* We note that  $h^*$  is monotone, since  $h$  is maximal monotone by Proposition 3.4 and Remark 2.3. Thus, due to the maximality of  $h$ , it suffices to prove  $h \subseteq h^*$ . For doing so, let  $(u, v), (x, y) \in h$ . Then

$$\begin{aligned} \langle y|u \rangle_{\mathcal{BD}(G)} &= \langle \pi_{\mathcal{BD}(G)}^* y | \pi_{\mathcal{BD}(G)}^* u \rangle_E + \langle G \pi_{\mathcal{BD}(G)}^* y | G \pi_{\mathcal{BD}(G)}^* u \rangle_H \\ &= \langle \pi_{\mathcal{BD}(G)}^* y | \pi_{\mathcal{BD}(G)}^* u \rangle_E + \langle \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y | L \pi_{\mathcal{BD}(G)}^* u \rangle_H \\ &= \langle \pi_{\mathcal{BD}(G)}^* y | \pi_{\mathcal{BD}(G)}^* u \rangle_E + \left( L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y - K^\diamond K \pi_{\mathcal{BD}(G)}^* x \right) (\pi_{\mathcal{BD}(G)}^* u) + \\ &\quad + \langle K \pi_{\mathcal{BD}(G)}^* x | K \pi_{\mathcal{BD}(G)}^* u \rangle_U. \end{aligned}$$

Using that  $\pi_{\mathcal{BD}(G)}^* x \in E_0$  and  $L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y - K^\diamond K \pi_{\mathcal{BD}(G)}^* x \in E$  we have  $L^\diamond \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y - K^\diamond K \pi_{\mathcal{BD}(G)}^* x = -D \pi_{\mathcal{BD}(D)}^* \overset{\bullet}{G} y = -\pi_{\mathcal{BD}(D)}^* y$  according to Lemma 3.1. Thus,

$$\langle y|u \rangle_{\mathcal{BD}(G)} = \langle K \pi_{\mathcal{BD}(G)}^* x | K \pi_{\mathcal{BD}(G)}^* u \rangle_U.$$

Repeating this argumentation and interchanging  $y$  and  $x$  as well as  $u$  and  $v$ , we get that

$$\langle v|x \rangle_{\mathcal{BD}(G)} = \langle K \pi_{\mathcal{BD}(G)}^* u | K \pi_{\mathcal{BD}(G)}^* x \rangle_U$$

and hence,  $\langle y|u \rangle_{\mathcal{BD}(G)} = \langle x|v \rangle_{\mathcal{BD}(G)}$ , which implies  $h \subseteq h^*$ .  $\square$

This completes the proof of (i) $\Rightarrow$ (ii) in Theorem 3.2. To show the converse implication, we need the following well-known result for selfadjoint relations, which for sake of completeness, will be proved.

**Theorem 3.6** ([1, Theorem 5.3]). *Let  $Y$  be a Hilbert space and  $C \subseteq Y \oplus Y$  a selfadjoint relation. Let  $U := \overline{[Y]C}$ . Then there exists a selfadjoint operator  $S : [Y]C \subseteq U \rightarrow U$  such that*

$$C = S \oplus \left( \{0\} \times U^\perp \right).$$

*Proof.* Due to the selfadjointness of  $C$ , we have that

$$U = \overline{[Y]C} = \overline{[Y]C^*} = (C[\{0\}])^\perp. \quad (5)$$

We define the relation  $S := \{(u, v) \in U \oplus U \mid (u, v) \in C\}$  and prove that  $S$  is a mapping. First we note that  $S$  is linear as  $C$  and  $U$  are linear. Thus, it suffices to show that  $(0, v) \in S$  for some  $v \in U$  implies  $v = 0$ . Indeed, if  $(0, v) \in S$ , we have  $(0, v) \in C$  and hence,  $v \in C[\{0\}] = U^\perp$ . Thus,  $v \in U \cap U^\perp = \{0\}$  and hence,  $S$  is a mapping. Next, we show that  $C = S \oplus (\{0\} \times U^\perp)$ . First note that  $S \subseteq C$  as well as  $\{0\} \times U^\perp \subseteq C$  by definition and hence,  $S \oplus (\{0\} \times U^\perp) \subseteq C$  due to the linearity of  $C$ . Let now  $(u, v) \in C$  and decompose  $v = v_0 + v_1$  for  $v_0 \in U, v_1 \in U^\perp = C[\{0\}]$ . Hence,  $(0, v_1) \in C$  and  $(u, v_0) = (u, v) - (0, v_1) \in C$ . Moreover,  $u \in U$  by (5) and thus, we derive  $(u, v_0) \in S$  and consequently,  $(u, v) = (u, v_0) + (0, v_1) \in S \oplus (\{0\} \times U^\perp)$ . Finally, we show that  $S$  is selfadjoint. Using that  $C = S \oplus (\{0\} \times U^\perp)$ , we obtain that

$$\begin{aligned} (x, y) \in S^* &\Leftrightarrow x, y \in U \wedge \forall (u, v_0) \in S : \langle v_0|x \rangle_U = \langle u|y \rangle_U \\ &\Leftrightarrow x, y \in U \wedge \forall (u, v_0) \in S, v_1 \in U^\perp : \langle v_0 + v_1|x \rangle_Y = \langle v_0|x \rangle_Y = \langle u|y \rangle_Y \\ &\Leftrightarrow x, y \in U \wedge \forall (u, v) \in C : \langle v|x \rangle_Y = \langle u|y \rangle_Y \\ &\Leftrightarrow x, y \in U \wedge (x, y) \in C^* = C \\ &\Leftrightarrow (x, y) \in S, \end{aligned}$$

which gives  $S = S^*$ , i.e.  $S$  is selfadjoint.  $\square$

*Remark 3.7.* It is obvious that in case of a monotone selfadjoint relation  $C$  in the latter theorem, the operator  $S$  is monotone, too.

**Lemma 3.8.** *Assume (ii) in Theorem 3.2 and let  $h = S \oplus (\{0\} \times U^\perp)$  with  $U := \overline{[\mathcal{BD}(G)]h}$  and  $S : [\mathcal{BD}(G)]h \subseteq U \rightarrow U$  selfadjoint. We define the vectorspace*

$$E_0 := \left\{ u \in \mathcal{D}(G) \mid \pi_{\mathcal{BD}(G)}u \in \mathcal{D}(\sqrt{S}) \right\} \subseteq E$$

*and the operators  $L : E_0 \rightarrow H, w \mapsto Gw$  and  $K : E_0 \rightarrow U, u \mapsto \sqrt{S}\pi_{\mathcal{BD}(G)}u$ . Then the operator  $\begin{pmatrix} L \\ K \end{pmatrix} : E_0 \subseteq E \rightarrow H \oplus U$  is closed and  $E_0, U, L$  and  $K$  satisfy the hypothesis, if we equip  $E_0$  with the graph norm of  $\begin{pmatrix} L \\ K \end{pmatrix}$ .*

*Proof.* Let  $(w_n)_{n \in \mathbb{N}}$  be a sequence in  $E_0$  such that  $w_n \rightarrow w$  in  $E$ ,  $Gw_n = Lw_n \rightarrow v$  in  $H$  and  $\sqrt{S}\pi_{\mathcal{BD}(G)}w_n = Kw_n \rightarrow z$  in  $U$  for some  $w \in E, v \in H, z \in U$ . Due to the closedness of  $G$  we infer that  $w \in \mathcal{D}(G)$  and  $v = Gw$ . Thus,  $(w_n)_{n \in \mathbb{N}}$  converges to  $w$  in  $\mathcal{D}_G$  and hence,  $\pi_{\mathcal{BD}(G)}w_n \rightarrow \pi_{\mathcal{BD}(G)}w$ . By the closedness of  $\sqrt{S}$ , we get  $\pi_{\mathcal{BD}(G)}w \in \mathcal{D}(\sqrt{S})$  and  $z = \sqrt{S}\pi_{\mathcal{BD}(G)}w$ . Thus,  $w \in E_0$  and  $\begin{pmatrix} L \\ K \end{pmatrix} w = \begin{pmatrix} v \\ z \end{pmatrix}$  and hence,  $\begin{pmatrix} L \\ K \end{pmatrix}$  is closed. Thus,  $E_0$  equipped with the graph norm of  $\begin{pmatrix} L \\ K \end{pmatrix}$  is a Hilbert space. Moreover,  $E_0, U, L$  and  $K$  satisfy the hypothesis, since clearly  $\mathcal{D}(G_0) \subseteq E_0 \subseteq \mathcal{D}(G)$ , which gives  $E_0 \subseteq E$  dense and  $G_0 \subseteq L \subseteq G$ . Moreover, by definition we have  $K|_{\mathcal{D}(G_0)} = 0$ .  $\square$

The only thing, which is left to show, is that  $\mathcal{D}(A)$  is given as in Theorem 3.2 (i).

**Lemma 3.9.** *Assume that (ii) in Theorem 3.2 holds and let  $E_0, U$  and  $K, L$  be as in Lemma 3.8. Then*

$$\mathcal{D}(A) = \{(u, w) \in E_0 \times H \mid K^\diamond Ku - L^\diamond w \in E\}.$$

*Proof.* Let  $(u, w) \in \mathcal{D}(A)$ , i.e.  $(u, w) \in \mathcal{D}(G) \times \mathcal{D}(D)$  with  $(\pi_{\mathcal{BD}(G)}u, \dot{D}\pi_{\mathcal{BD}(D)}w) \in h$ . Then, by definition of  $U$  and  $S$ , we have that  $\pi_{\mathcal{BD}(G)}u \in \mathcal{D}(S) \subseteq E_0$  and  $\dot{D}\pi_{\mathcal{BD}(D)}w - S\pi_{\mathcal{BD}(G)}u \in U^\perp$ . Let  $x \in E_0$  and set  $w_0 := w - P_{\mathcal{BD}(D)}w \in \mathcal{D}(D_0)$  as well as  $x_0 := x - P_{\mathcal{BD}(G)}x \in \mathcal{D}(G_0)$ .

Then we compute

$$\begin{aligned}
(K^\diamond Ku - L^\diamond w)(x) &= \langle Ku | Kx \rangle_U - \langle w | Lx \rangle_H \\
&= \langle \sqrt{S} \pi_{\mathcal{BD}(G)} u | \sqrt{S} \pi_{\mathcal{BD}(G)} x \rangle_U - \langle w | Gx \rangle_H \\
&= \langle S \pi_{\mathcal{BD}(G)} u - \dot{D} \pi_{\mathcal{BD}(D)} w | \pi_{\mathcal{BD}(G)} x \rangle_{\mathcal{BD}(G)} + \\
&\quad + \langle \dot{D} \pi_{\mathcal{BD}(D)} w | \pi_{\mathcal{BD}(G)} x \rangle_{\mathcal{BD}(G)} - \langle w | Gx \rangle_H \\
&= \langle P_{\mathcal{BD}(D)} w | GP_{\mathcal{BD}(G)} x \rangle_H + \langle DP_{\mathcal{BD}(G)} w | P_{\mathcal{BD}(G)} x \rangle_E - \langle w | Gx \rangle_H \\
&= \langle P_{\mathcal{BD}(D)} w | GP_{\mathcal{BD}(G)} x \rangle_H + \langle DP_{\mathcal{BD}(G)} w | P_{\mathcal{BD}(G)} x \rangle_E - \\
&\quad - \langle w | GP_{\mathcal{BD}(G)} x \rangle_H - \langle w | G_0 x_0 \rangle_H \\
&= \langle -w_0 | GP_{\mathcal{BD}(G)} x \rangle_H + \langle DP_{\mathcal{BD}(G)} w | P_{\mathcal{BD}(G)} x \rangle_E + \langle Dw | x_0 \rangle_E \\
&= \langle D_0 w_0 | P_{\mathcal{BD}(G)} x \rangle_E + \langle DP_{\mathcal{BD}(G)} w | P_{\mathcal{BD}(G)} x \rangle_E + \langle Dw | x_0 \rangle_E \\
&= \langle Dw | x \rangle_E,
\end{aligned}$$

where we have used  $\pi_{\mathcal{BD}(G)} x \in \mathcal{D}(\sqrt{S}) \subseteq U$  in the fourth equality. Thus,  $K^\diamond Ku - L^\diamond w = Dw \in E$ . Moreover,  $u \in E_0$  since  $\pi_{\mathcal{BD}(G)} u \in \mathcal{D}(S) \subseteq \mathcal{D}(\sqrt{S})$ . This proves one inclusion. Let now  $(u, w) \in E_0 \times H$  with  $K^\diamond Ku - L^\diamond w \in E$ . Then by Lemma 3.1  $w \in \mathcal{D}(D)$  with  $K^\diamond Ku - L^\diamond w = Dw$ . We need to prove that  $\pi_{\mathcal{BD}(G)} u \in \mathcal{D}(S)$ . We already have  $\pi_{\mathcal{BD}(G)} u \in \mathcal{D}(\sqrt{S})$ , by definition of  $E_0$ . Let now  $v \in \mathcal{D}(\sqrt{S})$ . Then we have

$$\begin{aligned}
\langle \sqrt{S} \pi_{\mathcal{BD}(G)} u | \sqrt{S} v \rangle_U &= \langle Ku | K \pi_{\mathcal{BD}(G)}^* v \rangle_U \\
&= (K^\diamond Ku - L^\diamond w) (\pi_{\mathcal{BD}(G)}^* v) + \langle w | L \pi_{\mathcal{BD}(G)}^* v \rangle_H \\
&= \langle Dw | \pi_{\mathcal{BD}(G)}^* v \rangle_E + \langle w | G \pi_{\mathcal{BD}(G)}^* v \rangle_H \\
&= \langle w | G \pi_{\mathcal{BD}(G)}^* v \rangle_{\mathcal{D}_D} \\
&= \langle \pi_{\mathcal{BD}(D)} w | \dot{G} v \rangle_{\mathcal{BD}(D)} \\
&= \langle \dot{D} \pi_{\mathcal{BD}(D)} w | v \rangle_U,
\end{aligned}$$

which gives  $\pi_{\mathcal{BD}(G)} u \in \mathcal{D}(S)$  with  $S \pi_{\mathcal{BD}(G)} u = \dot{D} \pi_{\mathcal{BD}(D)} w$ . Hence,  $(\pi_{\mathcal{BD}(G)} u, \dot{D} \pi_{\mathcal{BD}(D)} w) \in S \subseteq h$ , and thus,  $(u, w) \in \mathcal{D}(A)$ .  $\square$

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